ATKINSON SCIENCE

THEORY GUIDE

# Conical Shock Web Application

Keith Atkinson

21 September 2020

Atkinson Science welcomes your comments on this Theory Guide. Please send an email to keith.atkinson@atkinsonscience.co.uk.

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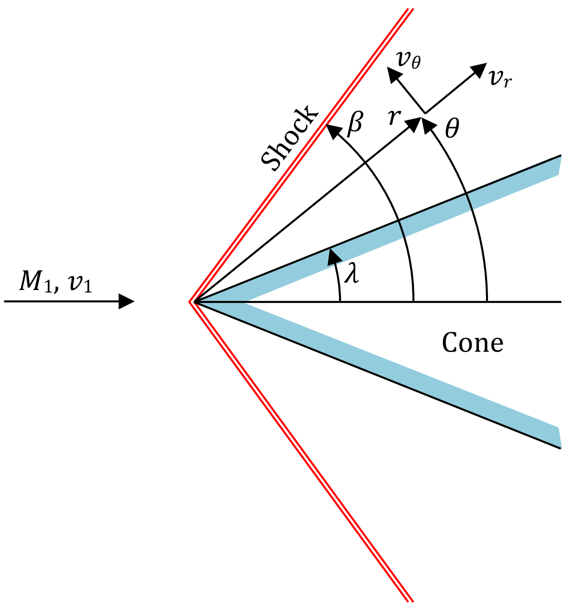
# 1 Introduction

You can find the Atkinson Science Conical Shock web application at the web address https://atkinsonscience.co.uk/WebApps/Aerospace/ConicalShock.aspx. You can also find a user guide at https://atkinsonscience.co.uk/PDFs/WebApps/Conical%20Shock%20User%20Guide.pdf.

A conical shock is formed when air at supersonic speed approaches a cone parallel to the axis of the cone, as shown in Figure 1. We shall assume that the cone extends to infinity in the downstream direction (the cone is semi-infinite). A conical shock is formed at the vertex of the cone. The change in properties across the shock can be assumed to be the same as for an oblique shock formed at the vertex of a sharp wedge. However, the wave angle $\beta$ of the conical shock will be less than that of the oblique shock, assuming the semi-vertex angle $\lambda$ of the cone and the deflection angle $\alpha$ of the wedge are the same.

We can define a spherical coordinate system $(r, \theta, \phi)$ with origin at the vertex of the cone, as shown in Figure 1. The flow is axisymmetric, so there are no variations in flow properties in the $\phi$ direction. We can define velocity components $v_r$ and $v_\theta$ along the $r$ and $\theta$ directions, respectively. Since the flow is not swirling, the component of velocity $v_\phi$ in the $\phi$ direction is zero.

**Figure 1 Cone and spherical coordinate system**



The flow behind a conical shock differs from that behind an oblique shock. After traversing the conical shock, the flow streamlines curve until they become parallel with the surface of the cone at infinity. In contrast, the streamlines behind an oblique shock become parallel to the surface of the wedge immediately. Since the cone extends to infinity the idea that flow properties may vary along the surface of the cone is not meaningful. In fact, experiments show that the flow properties are constant along the surface of the cone and along rays originating from the vertex. Variations in flow properties are only significant in the $\theta$ direction.

A method of solving the flow behind a conical shock has been published by Taylor and Maccoll, Ref. [1]. They assume that the flow behind the shock is irrotational. This assumption leads to the following simple relation between the velocity components $v_r$ and $v_\theta$ behind the shock:

$$v_\theta = \frac{dv_r}{d\theta} \qquad (1.1)$$

If $h_2$ and $v_2$ are the downstream static enthalpy and speed given by the oblique shock relations, then we can define a reference velocity $v_{max}$ as follows:

$$h_{02} = h_2 + \frac{v_2^2}{2} = \frac{v_{max}^2}{2} \qquad (1.2)$$

Note that $v_{max}$ is the same everywhere behind the shock. Taylor and Maccoll derive the following relation for the radial velocity component $v_r$ in terms of the angle $\theta$:

$$\frac{\gamma - 1}{2}\left[v_{max}^2 - v_r^2 - \left(\frac{dv_r}{d\theta}\right)^2\right]\left[2v_r + \frac{dv_r}{d\theta}\cot\theta + \frac{d^2v_r}{d\theta^2}\right] - \frac{dv_r}{d\theta}\left[v_r\frac{dv_r}{d\theta} + \frac{dv_r}{d\theta}\frac{d^2v_r}{d\theta^2}\right] = 0 \qquad (1.3)$$

This is the Taylor-Maccoll equation for the solution of conical shock flows. The derivation of the equation can also be found in text books on compressible fluid flow, such as Ref. [2].

There is no analytical solution satisfying equations (1.1) to (1.3). We must solve them numerically and the approach taken will be described in Sections 3 and 4 of this guide.

# 2 Shock characteristics

The conical shock formed around a cone has the characteristics of an oblique shock. Usually, it is a weak oblique shock that is formed, rather than a strong oblique shock. However, if the pressure at the base of the cone can be increased, then it is possible to create a conical shock with the characteristics of a strong oblique shock (just as when the pressure at the base of a wedge is increased). Both types of shock are attached to the vertex of the cone, but the changes in flow properties across the strong shock are more severe. The web application computes only the weak shock solution.

Figure 2 applies to the weak shock solution and shows how the wave angle $\beta$ varies with the semi-vertex angle $\lambda$ and the upstream Mach number $M_1$. The ratio of the specific heats $\gamma$ is assumed to be 1.4. For each Mach number $M_1$ there is a highest possible semi-vertex angle $\lambda_{max}$ and a highest possible wave angle $\beta_{max}$. Figure 3 shows $\lambda_{max}$ against the $M_1$, and Figure 4 shows the $\beta_{max}$ against $M_1$.

**Figure 2 Wave angle against semi-vertex angle for different Mach number values**

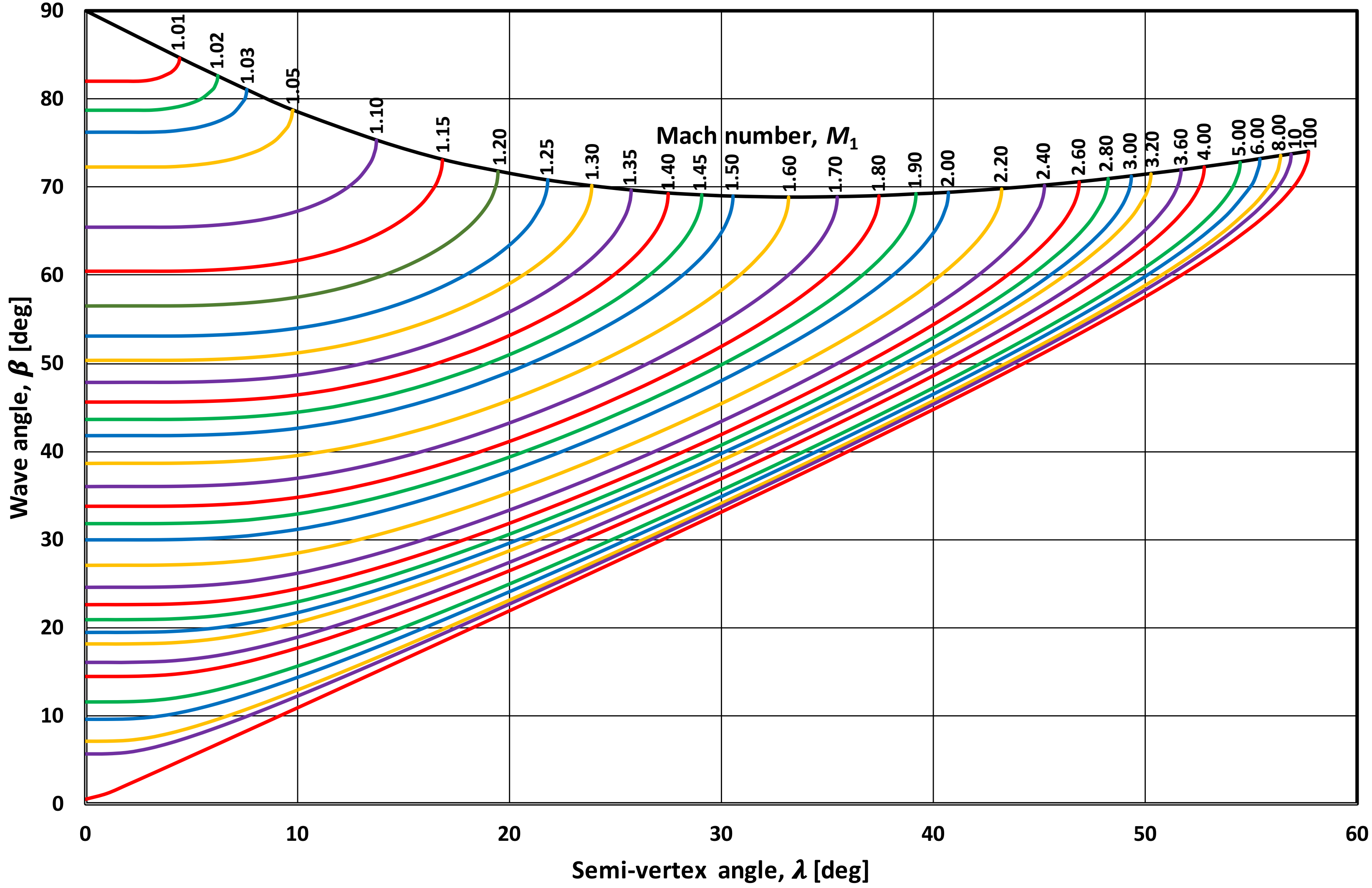


**Figure 3 Highest semi-vertex angle for a given upstream Mach number**

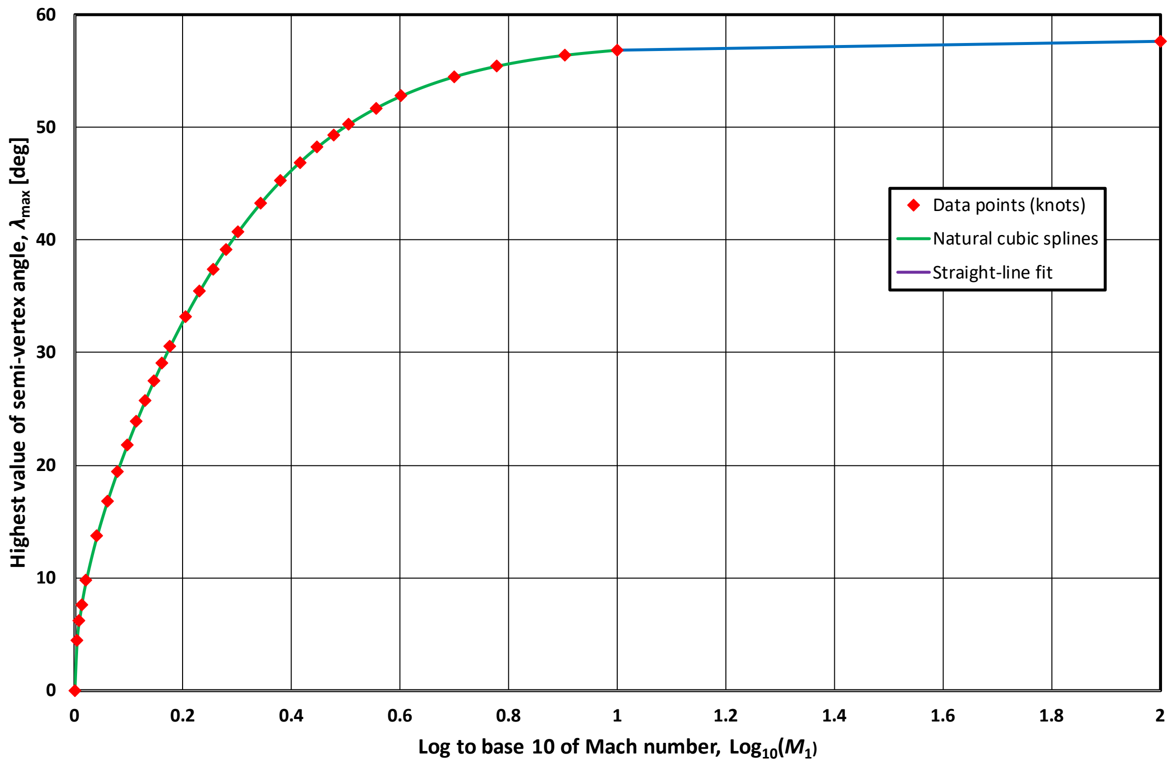
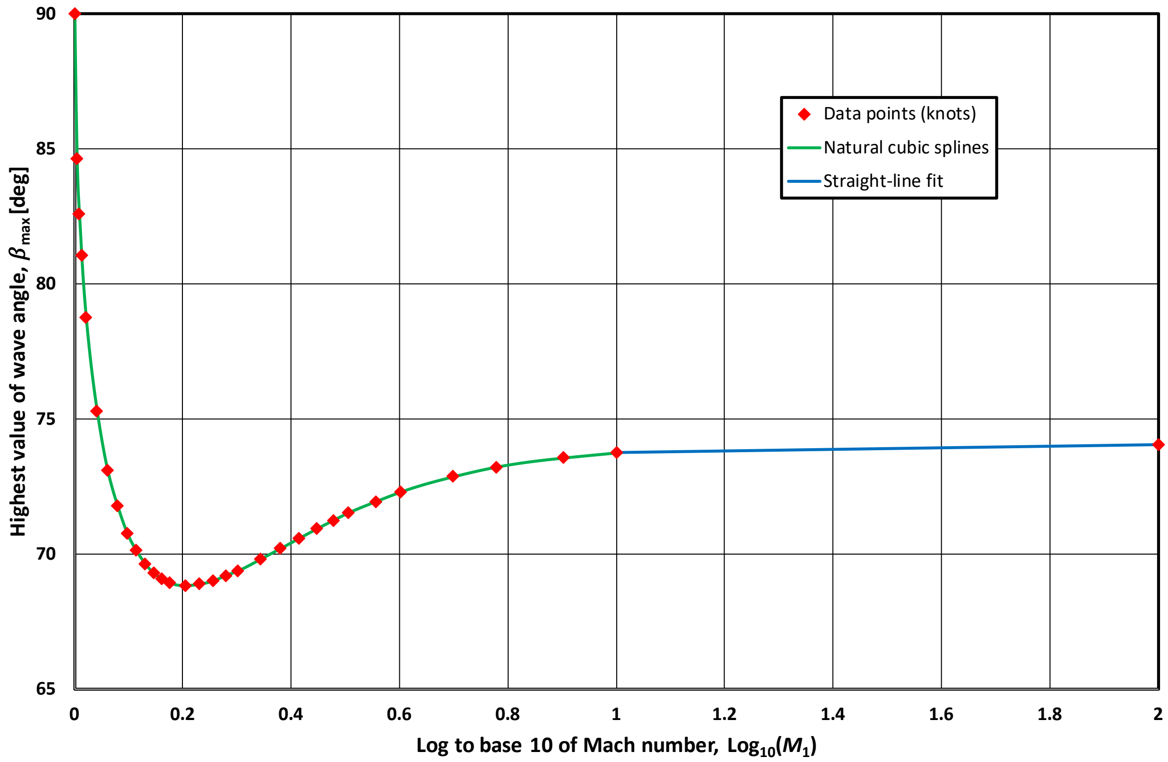
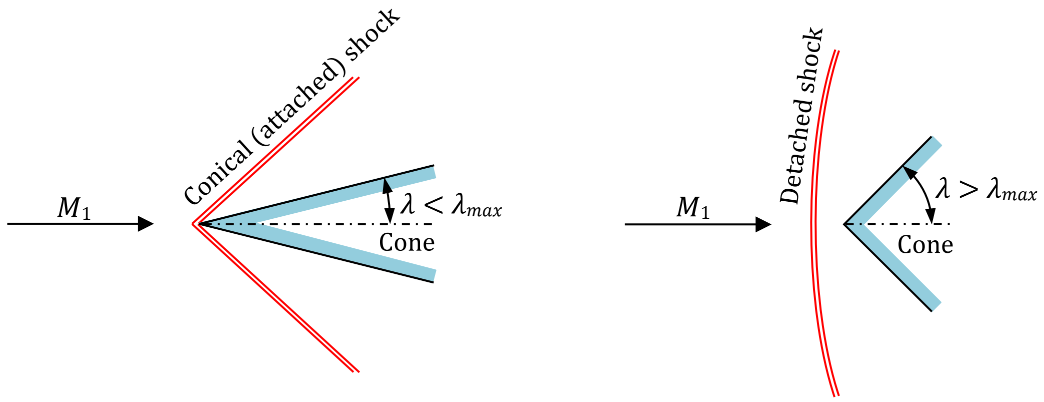



**Figure 4 Highest wave angle for a given upstream Mach number**



If the semi-vertex angle exceeds the $\lambda_{max}$ value for the upstream Mach number, then the shock becomes detached, as shown in Figure 5. The web application does not deal with the solution of the detached shock. The highest semi-vertex angle $\lambda_{max}$ at which a conical shock can be formed increases with $M_1$, as can be seen from Figure 3. The web application can determine whether the user's input would produce a detached shock. If this is the case, then the application issues a warning message and advises the user to either reduce the semi-vertex angle or increase the upstream Mach number.

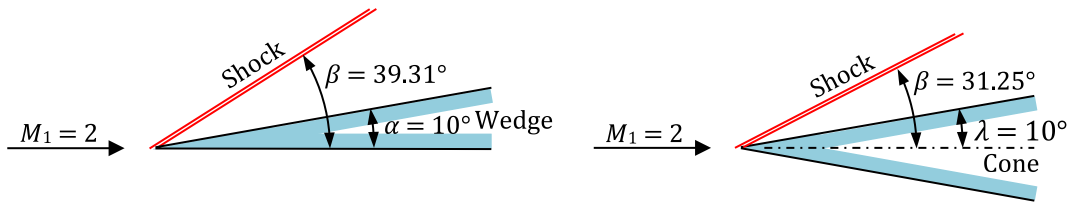
**Figure 5 Conical and detached shocks**




The wave angle $\beta$ of a conical shock for a cone with semi-vertex angle $\lambda$ is generally smaller than the wave angle of an oblique shock with a wedge deflection angle $\alpha$ equal to $\lambda$. However, as $\alpha$ and $\lambda$ approach zero, the wave angles of the two types of shock become the same.

Figure 6 shows an oblique shock and a conical shock for an upstream Mach number $M_1$ of 2. The angles $\alpha$ and $\lambda$ are the same at 10°. The wave angle for the oblique shock is 39.31°, while the wave angle for the conical shock is 31.25°. The wave angle of the conical shock only becomes equal to that of the oblique shock when $\lambda$ is increased to 21.38°.

**Figure 6 Difference in wave angle between oblique shock and conical shock**



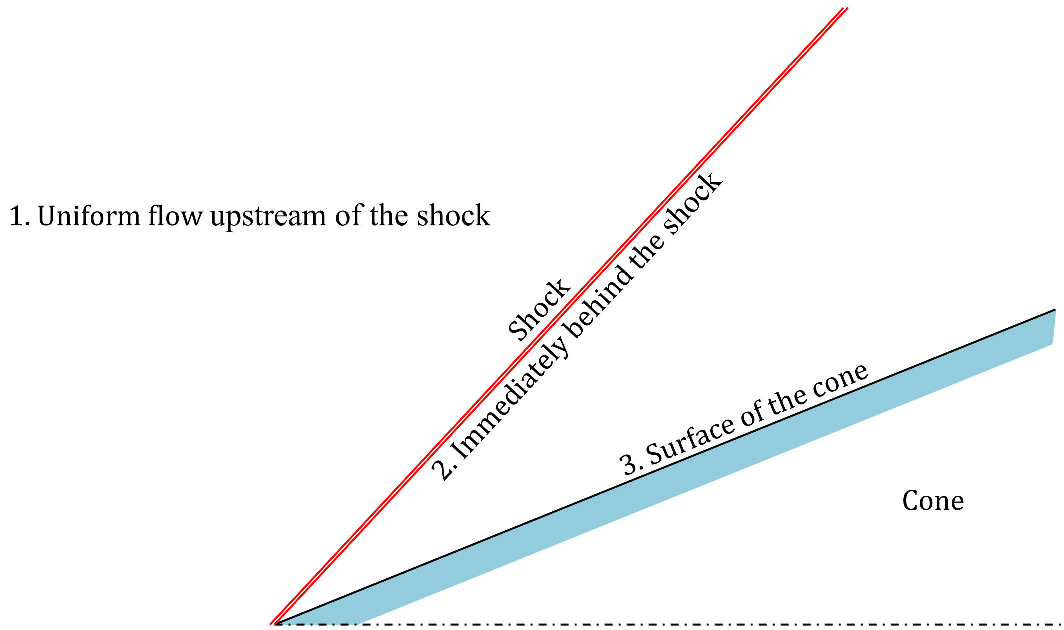
# 3 Solution of the flow

## 3.1 Calculation method

In order to describe the calculation method, we must distinguish between three regions of the flow, as shown in Figure 7. Region 1 is the uniform flow upstream of the shock. We shall base the properties of this flow on the International Standard Atmosphere (ISA), Ref. [3]. For the ISA, the ratio of specific heats $\gamma$ is equal to 1.4. Region 2 is the flow immediately behind the shock. If we know the wave angle $\beta$ then we can calculate the properties behind the shock from oblique shock relations. Region 3 is the surface of the cone. Between 2 and 3 we can assume that the flow is isentropic and calculate the flow from equations (1.1) to (1.3). To describe the calculation method, we shall use the subscripts 1, 2 and 3 to indicate the locations of flow properties.

At the start of the calculation we do not know the wave angle $\beta$. We must specify an initial value of $\beta$ so that we can determine the properties immediately behind the shock. We then divide the space between 2 and 3 into uniform angular intervals $\delta\theta$. Using the conditions behind the shock as boundary values, we solve equations (1.1) to (1.3) numerically in steps of $\delta\theta$, marching away from the shock towards the cone. If our initial value of $\beta$ is correct, then we will find that the calculated angular velocity component $v_\theta$ is zero at the cone. If not, then we must repeat the calculation for a different shock angle $\beta$, until the condition that $v_\theta$ is zero at the cone is satisfied.

**Figure 7 Flow regions**



## 3.2 Starting the calculation

Figure 2 shows that for a weak shock solution there is a highest possible semi-vertex angle $\lambda_{max}$ and a highest possible wave angle $\beta_{max}$ for the upstream Mach number $M_1$. The variation of $\lambda_{max}$ with $M_1$ is plotted in Figure 3 and the variation of $\beta_{max}$ with $M_1$ is plotted in Figure 4. These plots were obtained by increasing the semi-vertex angle $\lambda$ entered into the web application for a given $M_1$ until it was no longer possible to obtain a weak shock solution. The increments in $\lambda$ were $0.001°$. In Table 3, the computed values of $\lambda_{max}$ and $\beta_{max}$ are given against $M_1$ and $\text{Log}_{10}M_1$.

**Table 1 $\lambda_{max}$ and $\beta_{max}$ against $M_1$**

| $M_1$ | $\text{Log}_{10}M_1$ | $\lambda_{max}$ [deg] | $\beta_{max}$ [deg] |
|---|---|---|---|
| 1 | 0 | 0 | 90 |
| 1.01 | 0.004321 | 4.448 | 84.641 |
| 1.02 | 0.008600 | 6.234 | 82.585 |
| 1.03 | 0.012837 | 7.598 | 81.049 |
| 1.05 | 0.021189 | 9.759 | 78.743 |
| 1.1 | 0.041393 | 13.749 | 75.290 |
| 1.15 | 0.060698 | 16.846 | 73.113 |
| 1.2 | 0.079181 | 19.475 | 71.782 |
| 1.25 | 0.096910 | 21.792 | 70.777 |
| 1.3 | 0.11394 | 23.874 | 70.123 |
| 1.35 | 0.13033 | 25.767 | 69.636 |
| 1.4 | 0.14613 | 27.499 | 69.295 |
| 1.45 | 0.16137 | 29.092 | 69.083 |
| 1.5 | 0.17609 | 30.563 | 68.943 |
| 1.6 | 0.20412 | 33.189 | 68.835 |
| 1.7 | 0.23045 | 35.459 | 68.889 |
| 1.8 | 0.25527 | 37.436 | 69.008 |
| 1.9 | 0.27875 | 39.166 | 69.205 |
| 2 | 0.30103 | 40.690 | 69.380 |
| 2.2 | 0.34242 | 43.234 | 69.807 |
| 2.4 | 0.38021 | 45.258 | 70.211 |
| 2.6 | 0.41497 | 46.894 | 70.591 |
| 2.8 | 0.44716 | 48.232 | 70.938 |
| 3 | 0.47712 | 49.340 | 71.247 |
| 3.2 | 0.50515 | 50.268 | 71.524 |
| 3.6 | 0.55630 | 51.719 | 71.947 |
| 4 | 0.60206 | 52.787 | 72.300 |
| 5 | 0.69897 | 54.482 | 72.863 |
| 6 | 0.77815 | 55.433 | 73.219 |
| 8 | 0.90309 | 56.402 | 73.573 |
| 10 | 1 | 56.860 | 73.744 |
| 100 | 2 | 57.678 | 74.037 |

We fitted natural cubic splines to the data $\lambda_{max}$ vs. $\text{Log}_{10}M_1$ and the data $\beta_{max}$ vs. $\text{Log}_{10}M_1$ from $\text{Log}_{10}1$ to $\text{Log}_{10}10$ to enable the web application to interpolate between the tabulated values of $M_1$. We simply joined the points at $\text{Log}_{10}10$ and $\text{Log}_{10}100$ with a straight line, so that $\lambda_{max}$ and $\beta_{max}$ could be interpolated up to $M_1 = 100$. The curve-fits can be seen in Figures 3 and 4.

Before starting a calculation, the web application uses the $\lambda_{max}$ vs. $\text{Log}_{10}M_1$ curve-fit to check that the $\lambda$ value entered does not exceed $\lambda_{max}$ for the $M_1$ value entered. If it does, then the application issues a warning message and advises the user to either reduce $\lambda$ or increase $M_1$.

To start the calculation, the web application uses the $\beta_{max}$ vs. $\text{Log}_{10}M_1$ curve-fit to set the initial guess for $\beta$ to the value of $\beta_{max}$ for the $M_1$ value entered. As the calculation proceeds, $\beta$ is reduced step-by-step from $\beta_{max}$ until the condition that $v_\theta$ is zero at the cone is satisfied (unless $\lambda = \lambda_{max}$ at $M_1$ is entered by the user, in which case $\beta_{max}$ is the required wave angle).

## 3.3 Properties ahead of the shock

The properties of the flow upstream of the shock are based on the International Standard Atmosphere (ISA), Ref. [3]. The ISA is a model of the change in temperature and pressure with altitude in the Earth's atmosphere. The atmosphere is divided into layers over which the temperature is either constant or varies linearly with geopotential altitude, as shown in Figure 8. Ref. [3] defines constants and formulae by which the pressure and other properties of the atmosphere may be calculated from the temperature. The constants are set out in Table 2.

**Figure 8 Temperature of the International Standard Atmosphere**

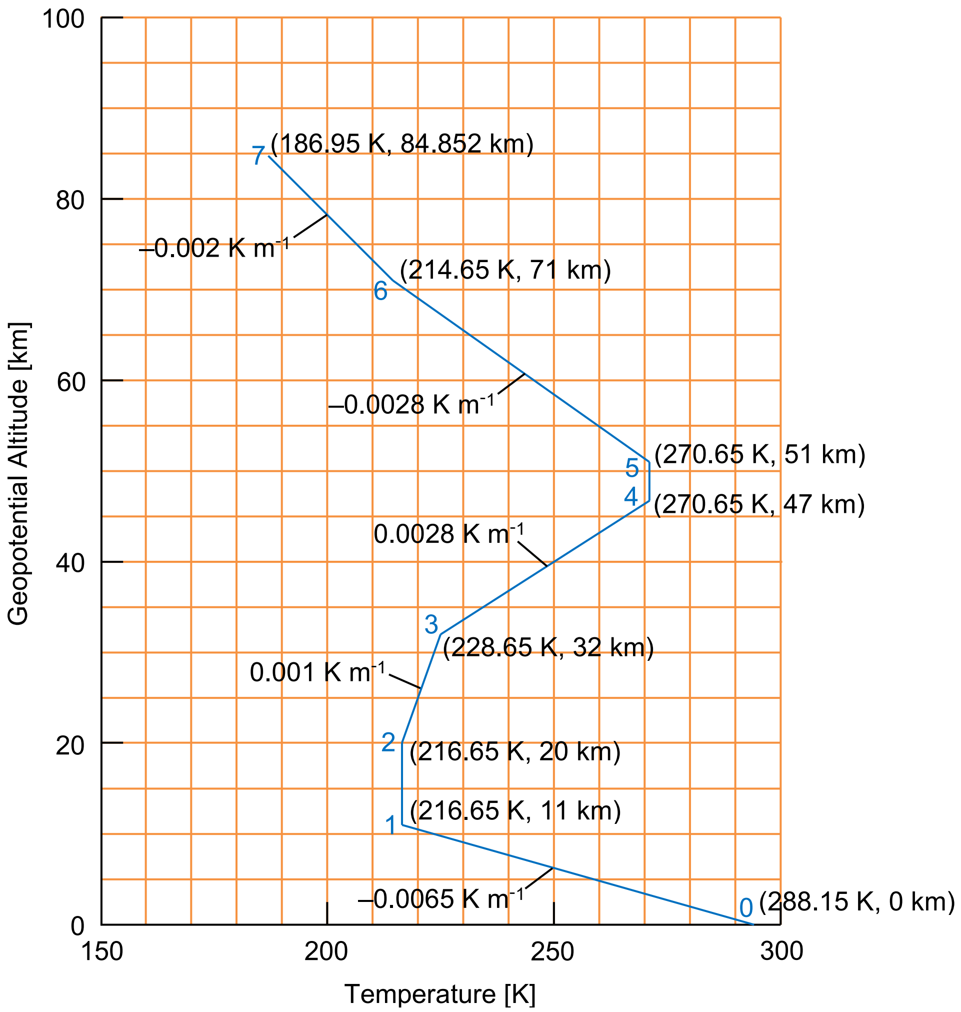


**Table 2 Properties of the International Standard Atmosphere**

| Standard values at sea level | |
|---|---|
| Temperature $T$ | 288.15 K |
| Pressure $p$ | 101,325 Pa |
| Density $\rho$ | 1.2250 kg m$^{-3}$ |
| Dynamic viscosity $\mu$ | 1.7894 × 10$^{-5}$ kg m$^{-1}$ s$^{-1}$ |
| Speed of sound $a$ | 340.29 m s$^{-1}$ |
| Acceleration due to gravity $g$ | 9.80665 m s$^{-2}$ |
| **Other standard values** | |
| Specific gas constant of air $R_{Air}$ | 287.05287 J kg$^{-1}$ K$^{-1}$ |
| Ratio of specific heats $\gamma = c_p/c_v$ | 1.4 |

The geopotential altitude $h$ is related to the geometric altitude $z$ by

$$h(z) = \left(\frac{R_E}{R_E + z}\right) z$$

where $R_E$ is the radius of the Earth (6,356 km). By rearranging this equation, we can write the geometric altitude $z$ in terms of the geopotential altitude $h$:

$$z(h) = \left(\frac{R_E}{R_E - h}\right) h$$

Ref. [3] shows how the atmospheric pressure and density are calculated given the variation in atmospheric temperature with altitude. The calculation steps are also given in Ref. [4]. Table 3 gives the pressure and density at the points 0 to 7 in Figure 8.

**Table 3 Pressure and density of the International Standard Atmosphere**

| Point | Geopotential altitude $h$ [m] | Geometric altitude $z$ [m] | Temperature $T$ [K] | Pressure $p$ [Pa] | Density $\rho$ [kg m$^{-3}$] |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 288.15 | 101,325 | 1.2250 |
| 1 | 11,000 | 11,109 | 216.65 | 22,632 | 0.3639 |
| 2 | 20,000 | 20,063 | 216.65 | 5,475 | 0.08804 |
| 3 | 32,000 | 32.162 | 228.65 | 868.0 | 0.01322 |
| 4 | 47,000 | 47.350 | 270.65 | 110.9 | 0.001427 |
| 5 | 51,000 | 51,413 | 270.65 | 66.94 | 8.616 × 10$^{-4}$ |
| 6 | 71,000 | 71,802 | 214.65 | 3.956 | 6.421 × 10$^{-5}$ |
| 7 | 84,852 | 86,000 | 186.95 | 0.3734 | 6.958 × 10$^{-6}$ |

The speed of sound upstream of the shock $a_1$ is given by

$$a_1 = \sqrt{\gamma R_{Air} T_1}$$

where the ratio of the specific heats $\gamma = C_p/C_v$ is defined to be constant and equal to 1.4 and the specific gas constant of the air $R_{Air}$ is defined to be 287.05287 J kg$^{-1}$ K$^{-1}$ (see Table 2).

The Mach number $M_1$ is

$$M_1 = \frac{v_1}{a_1}$$

where $v_1$ is the air speed.

The equations that relate the conditions on the two sides of an oblique shock assume that the air is calorically perfect. The static enthalpy $h_1$ is then equal to $C_p\, T_1$, where the specific heat at constant pressure $C_p$ is constant. In the web application $C_p$ is taken to be 1.004 kJ kg$^{-1}$ K$^{-1}$, which is the specific heat at constant pressure of dry air at 15°C.

If the air is assumed to be a calorically perfect gas, then the entropy of the air is given by

$$s_1 - s_o = C_p \ln(T_1/T_o) - R_{Air} \ln(p_1/p_o)$$

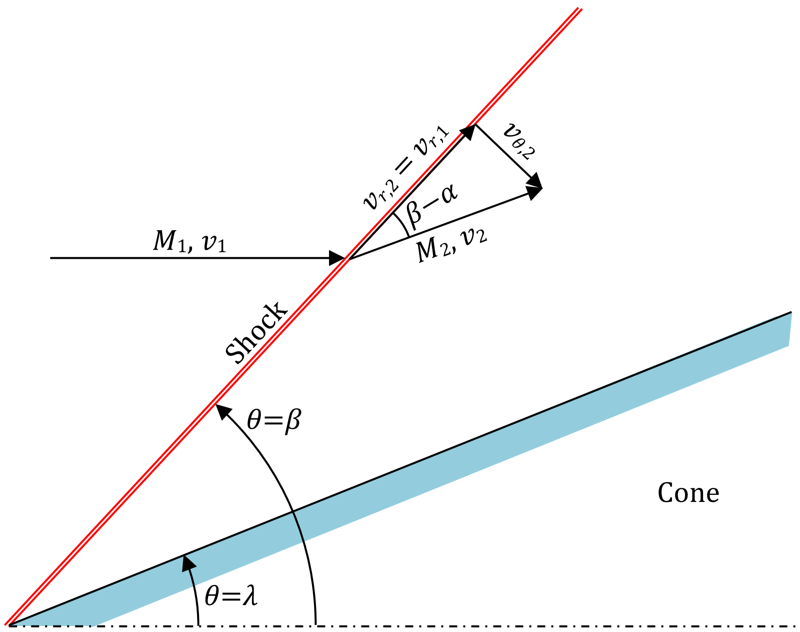
where ($p_o$, $T_o$, $s_o$) are conditions at some reference state. Referring to the tables of thermodynamic properties, Ref. [5], we have taken $s_o$ to be 6.86305 kJ kg$^{-1}$ K$^{-1}$ at 15°C and 1 bar pressure.

## 3.4 Properties immediately behind the shock

We do not know the wave angle $\beta$ of the conical shock. We must make an initial guess at $\beta$. If we guess correctly then when we have solved the flow behind the shock, the velocity component $v_\theta$ will be zero when $\theta = \lambda$, the semi-vertex angle of the cone. If it is not, then we must adjust $\beta$ until it is.

Figure 9 shows the velocity components immediately behind the shock. The shock lies along a radial line, so the velocity components and thermodynamic properties are constant along the shock. We can use equations developed for an oblique shock to calculate the velocity components and properties.

**Figure 9 Velocity components immediately behind the shock**




For the upstream Mach number $M_1$ and the guessed value of $\beta$, we can calculate the wedge deflection angle $\alpha$ from the oblique shock relation:

$$\tan\alpha = 2\cot\beta \frac{M_1^2 \sin^2\beta - 1}{M_1^2(\gamma + \cos 2\beta) + 2}$$

The Mach number $M_2$ immediately behind the shock is then given by

$$M_2^2 \sin^2(\beta - \alpha) = \frac{1 + \frac{\gamma - 1}{2} M_1^2 \sin^2\beta}{\gamma M_1^2 \sin^2\beta - \frac{\gamma - 1}{2}}$$

The pressure $p_2$, density $\rho_2$, temperature $T_2$, speed of sound $a_2$, static enthalpy $h_2$ and entropy $s_2$ immediately behind the shock can be calculated from

$$\frac{p_2 - p_1}{p_1} = \frac{2\gamma}{\gamma + 1}(M_1^2 \sin^2\beta - 1)$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1) M_1^2 \sin^2\beta}{(\gamma - 1) M_1^2 \sin^2\beta + 2}$$

$$\frac{T_2}{T_1} = \frac{h_2}{h_1} = \frac{a_2^2}{a_1^2} = 1 + \frac{2(\gamma - 1)}{(\gamma + 1)^2} \frac{M_1^2 \sin^2\beta - 1}{M_1^2 \sin^2\beta} (\gamma M_1^2 \sin^2\beta + 1)$$

$$\frac{s_2 - s_1}{R_{\mathrm{Air}}} = \ln\left\{\left[1 + \frac{2\gamma}{\gamma + 1}(M_1^2 \sin^2\beta - 1)\right]^{1/(\gamma - 1)} \left[\frac{(\gamma + 1) M_1^2 \sin^2\beta}{(\gamma - 1) M_1^2 \sin^2\beta + 2}\right]^{-\gamma/(\gamma - 1)}\right\}$$

The flow speed $v_2$ immediately behind the shock is given by

$$v_2 = M_2 a_2$$

The radial velocity component $v_{r,2}$ is

$$v_{r,2} = v_2 \cos(\beta - \alpha)$$

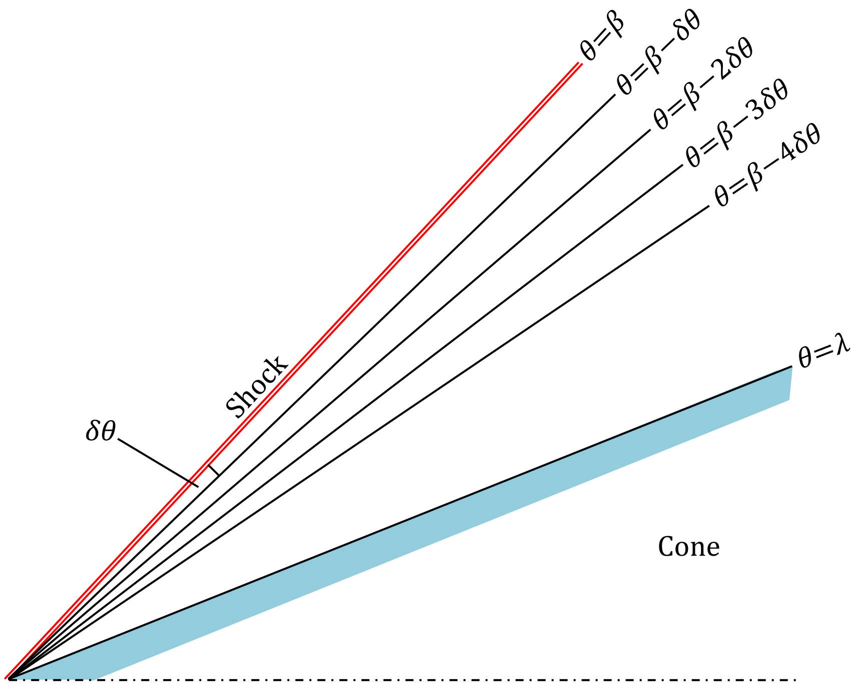
and angular velocity component $v_{\theta,2}$ is

$$v_{\theta,2} = -v_2 \sin(\beta - \alpha)$$

## 3.5 Velocity components between the shock and the cone

The space between the shock and the cone is divided into uniform intervals $\delta\theta$, as shown in Figure 10.

**Figure 10 Computation intervals**




We can define

$$v'_r = \frac{v_r}{v_{max}}$$

and

$$v'_\theta = \frac{v_\theta}{v_{max}}$$

where $v_{max}$ is the reference velocity defined by Eq. (1.2).

If we divide the Taylor-Maccoll equation (1.3) through by ${v_{max}}^3$ we obtain

$$\frac{\gamma-1}{2}\left[1-v'^2_r-\left(\frac{dv_r}{d\theta}\right)^2\right]\left[2v'_r+\frac{dv'_r}{d\theta}\cot\theta+\frac{d^2v'_r}{d\theta^2}\right]-\frac{dv'_r}{d\theta}\left[v'_r\frac{dv'_r}{d\theta}+\frac{dv'_r}{d\theta}\frac{d^2v'_r}{d\theta^2}\right]=0$$

Then, if we define

$$w=\frac{dv'_r}{d\theta} \qquad (3.1)$$

and substitute $w$ into the Taylor-Maccoll equation, we obtain

$$\frac{\gamma-1}{2}[1-v'^2_r-y^2]\left[2v'_r+y\cot\theta+\frac{dw}{d\theta}\right]-y^2\left[v'_r+\frac{dw}{d\theta}\right]=0$$

We can rearrange this equation so that the gradient term $dw/d\theta$ is on the left of the equals sign:

$$\frac{dw}{d\theta}=\frac{w^2v'_r-\left(\frac{\gamma-1}{2}\right)[1-(v'_r)^2-w^2][2v'_r+w\cot\theta]}{\left(\frac{\gamma-1}{2}\right)[1-(v'_r)^2-w^2]-w^2} \qquad (3.2)$$

We can integrate this equation using a numerical method to obtain $w$ $(=v'_\theta)$ and $v'_r$ on each of the radial lines $\theta=\beta-\delta\theta$, $\theta=\beta-2\delta\theta$, ..., $\theta=\lambda$.

The web application uses the classical fourth-order Runge-Kutta method to integrate Eqs. (3.1) and (3.2). The details of the method and a sample calculation are given in Section 4.

## 3.6 Thermodynamic properties between the shock and the cone

Once we have found the correct value of $\beta$ we can calculate the thermodynamic properties at the computation points between the shock wave and the cone using relations for isentropic flow.

From Eq. (1.2) we have

$$h + \frac{v^2}{2} = \frac{v_{max}^2}{2} \qquad (3.3)$$

or

$$\frac{a^2}{\gamma - 1} + \frac{v^2}{2} = \frac{v_{max}^2}{2} \qquad (3.4)$$

Multiplying both sides by $2/v^2$ gives

$$\frac{2}{\gamma - 1}\left(\frac{a}{v}\right)^2 + 1 = \left(\frac{v_{max}}{v}\right)^2$$

or

$$\frac{2}{\gamma - 1}\left(\frac{1}{M}\right)^2 + 1 = \left(\frac{v_{max}}{v}\right)^2$$

and so

$$\frac{v}{v_{max}} = \left[\frac{2}{\gamma - 1}\left(\frac{1}{M}\right)^2 + 1\right]^{-1/2} \qquad (3.5)$$

Using this equation, we can calculate $M$ from $v$.

Immediately behind the shock, the total temperature $T_{02}$ is given by

$$C_p T_{02} = C_p T_2 + \frac{v_2^2}{2}$$

The flow downstream of the shock is isentropic so we can use the following relations for isentropic flow to calculate the temperature, pressure and density at each computation point.

$$\frac{T_{02}}{T} = 1 + \frac{\gamma - 1}{2} M^2 \qquad (3.6)$$

$$\frac{p_{02}}{p} = \left(\frac{\rho_{02}}{\rho}\right)^{\gamma} = \left(\frac{T_{02}}{T}\right)^{\gamma/(\gamma - 1)} \qquad (3.7)$$

We can use the last equation with $p = p_2$ and $\rho = \rho_2$ to calculate the total properties $p_{02}$ and $\rho_{02}$ immediately behind the shock. The total properties remain unchanged downstream. The pressure $p$ and density $\rho$ anywhere downstream are then given by

$$\frac{p_{02}}{p} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma/(\gamma-1)} \qquad (3.8)$$

and

$$\frac{\rho_{02}}{\rho} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{1/(\gamma-1)} \qquad (3.9)$$

## 3.7 Thermodynamic properties at the surface of the cone

We can calculate the thermodynamic properties at the surface of the cone using the relations for isentropic flow given in the preceding section.

We can substitute $v = v_{r3}$ into (3.3), (3.4) and (3.5) to obtain the enthalpy $h_3$, the speed of sound $a_3$, and the Mach number $M_3$ at the surface of the cone:

$$h_3 + \frac{v_{r3}^2}{2} = \frac{v_{max}^2}{2}$$

$$\frac{a_3^2}{\gamma - 1} + \frac{v_{r3}^2}{2} = \frac{v_{max}^2}{2}$$

$$\frac{v_{r3}}{v_{max}} = \left[\frac{2}{\gamma - 1}\left(\frac{1}{M_3}\right)^2 + 1\right]^{-1/2}$$

The entropy $s_3$ is simply

$$s_3 = s_2$$

The temperature $T_3$ is given by (3.6):

$$\frac{T_{02}}{T_3} = 1 + \frac{\gamma - 1}{2} M_3^2$$

and the pressure $p_3$ and density $\rho_3$ are given by (3.7):

$$\frac{p_{02}}{p_3} = \left(\frac{\rho_{02}}{\rho_3}\right)^{\gamma} = \left(\frac{T_{02}}{T_3}\right)^{\gamma/(\gamma-1)}$$

# 4 Runge-Kutta method

The web application uses the classical fourth-order Runge-Kutta (RK4) method to solve the Taylor-Maccoll equation. The RK4 method is a one-step method for solving first-order ordinary differential equations. The Taylor-Maccoll equation (1.3) is a second-order ordinary differential equation, but by introducing the variable $w$, we have reduced the equation to a system of two first-order ordinary differential equations, (3.1) and (3.2). A description of the application of the RK4 method to systems of first-order equations can be found in many mathematics text books, for example, Ref. [6].

For convenience, we will change the variable names in (3.1) and (3.2), so that $\theta$ becomes $x$, $v'_r$ becomes $y$, and $w$ $(= v'_\theta)$ becomes $z$. Thus, Eqs. (3.1) and (3.2) can be written:

$$y' = f(x, y, z) = z \qquad (4.1)$$

$$z' = g(x, y, z) = \frac{z^2 y - 0.2[1 - y^2 - z^2][2y + z\cot x]}{0.2[1 - y^2 - z^2] - z^2} \qquad (4.2)$$

To illustrate the application of the RK4 method we will solve Eqs. (4.1) and (4.2) for a cone with a semi-vertex angle $\lambda$ of 30° moving at a Mach number $M_1$ of 1.6 at a geopotential altitude of 11,000 m.

## 4.1 Upstream conditions

The temperature, pressure and density of the International Standard Atmosphere at a geopotential altitude of 11,000 m are given in Table 3 and are $T_1 = 216.65$ K, $p_1 = 22{,}632$ Pa, $\rho_1 = 0.3639$ kg m$^{-3}$, respectively. From Table 1, the highest wave angle possible $\beta_{max}$ for a weak shock at $M_1 = 1.6$ is 68.835°. We shall use this as the starting value for $\beta$. The initial condition for $x$ is therefore 68.835°. The specific enthalpy $h_1$ is

$$h_1 = C_p T_1 = 1.004 \times 216.65 = 217.5166 \text{ kJ kg}^{-1}$$

and the specific entropy $s_1$ is

$$s_1 = s_o + C_p \ln(T_1/T_o) - R_{Air} \ln(p_1/p_o)$$

$$= 6.86305 + 1.004 \ln\left(\frac{216.65}{288.15}\right) - 0.28705287 \ln\left(\frac{22{,}632}{100{,}000}\right)$$

$$= 7.0032159 \text{ kJ kg}^{-1} \text{ K}^{-1}$$

The speed of sound upstream of the shock $a_1$ is

$$a_1 = \sqrt{\gamma R_{Air} T_1} = \sqrt{1.4 \times 287.05287 \times 216.65} = 295.06949 \text{ m s}^{-1}$$

and the speed of the flow $v_1$ is

$$v_1 = M_1 a_1 = 1.6 \times 295.06949 = 472.11119 \text{ m s}^{-1}$$

## 4.2 Conditions immediately behind the shock (initial conditions)

The flow conditions immediately behind the shock wave are given by the relations for an oblique shock in Section 3.4.

The wedge deflection angle $\alpha$ is given by

$$\tan\alpha = 2\cot\beta \frac{M_1^2\sin^2\beta - 1}{M_1^2(\gamma + \cos 2\beta) + 2}$$

$$= 2\cot 68.835 \frac{1.6^2\sin^2 68.835 - 1}{1.6^2(1.4 + \cos[2 \times 68.835]) + 2} = 0.2572324$$

so

$$\alpha = 14.425583°$$

The Mach number $M_2$ immediately behind the shock is given by

$$M_2^2\sin^2(\beta - \alpha) = \frac{1 + \frac{\gamma - 1}{2}M_1^2\sin^2\beta}{\gamma M_1^2\sin^2\beta - \frac{\gamma - 1}{2}}$$

so

$$M_2^2 = \frac{1}{\sin^2(68.835 - 14.425584)} \times \frac{1 + \frac{1.4 - 1}{2}1.6^2\sin^2 68.835}{1.4 \times 1.6^2\sin^2 68.835 - \frac{1.4 - 1}{2}} = 0.7492882$$

and

$$M_2 = 0.8656143$$

The pressure $p_2$ immediately behind the shock is given by

$$\frac{p_2 - p_1}{p_1} = \frac{2\gamma}{\gamma + 1}(M_1^2\sin^2\beta - 1) = \frac{2 \times 1.4}{1.4 + 1}(1.6^2\sin^2 68.835 - 1) = 1.4306561$$

From Table 3, the atmospheric pressure $p_1$ at a geopotential altitude of 11,000 m is 22,632 Pa. The pressure $p_2$ is therefore 55,010.608 Pa.

The density $\rho_2$ immediately behind the shock is given by

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)M_1^2\sin^2\beta}{(\gamma - 1)M_1^2\sin^2\beta + 2} = \frac{(1.4 + 1)1.6^2\sin^2 68.835}{(1.4 - 1)1.6^2\sin^2 68.835 + 2} = 1.8484844$$

From Table 3, the density of the atmosphere $\rho_1$ at a geopotential altitude of 11,000 m is 0.3639 kg m$^{-3}$. The density $\rho_2$ is therefore 0.6726635 kg m$^{-3}$.

The temperature $T_2$, static enthalpy $h_2$ and speed of sound $a_2$ immediately behind the shock are given by

$$\frac{T_2}{T_1} = \frac{h_2}{h_1} = \frac{a_2^2}{a_1^2} = 1 + \frac{2(\gamma - 1)}{(\gamma + 1)^2} \frac{M_1^2 \sin^2\beta - 1}{M_1^2 \sin^2\beta} (\gamma M_1^2 \sin^2\beta + 1)$$

$$= 1 + \frac{2(1.4 - 1)}{(1.4 + 1)^2} \times \frac{1.6^2 \sin^2 68.835 - 1}{1.6^2 \sin^2 68.835} (1.4 \times 1.6^2 \sin^2 68.835 + 1) = 1.3149454$$

From Table 3, the atmospheric temperature $T_1$ at a geopotential altitude of 11,000 m is 216.65 K. The temperature $T_2$ is therefore 284.88292 K.

The static enthalpy $h_1$ upstream of the shock is 217.5166 kJ kg$^{-1}$, so the specific enthalpy $h_2$ immediately behind the shock is 286.02245 kJ kg$^{-1}$.

The speed of sound $a_1$ upstream of the shock is 295.06949 m s$^{-1}$, so the speed of sound $a_2$ immediately behind the shock is 338.35934 m s$^{-1}$.

The entropy $s_2$ immediately behind the shock is given by

$$\frac{s_2 - s_1}{R_{Air}} = \ln\left\{\left[1 + \frac{2\gamma}{\gamma + 1}(M_1^2 \sin^2\beta - 1)\right]^{1/(\gamma-1)} \left[\frac{(\gamma + 1)M_1^2 \sin^2\beta}{(\gamma - 1)M_1^2 \sin^2\beta + 2}\right]^{-\gamma/(\gamma-1)}\right\}$$

$$= \ln\left\{\left[1 + \frac{2 \times 1.4}{1.4 + 1}(1.6^2 \sin^2 68.835 - 1)\right]^{1/(1.4-1)} \left[\frac{(1.4 + 1)1.6^2 \sin^2 68.835}{(1.4 - 1)1.6^2 \sin^2 68.835 + 2}\right]^{-1.4/(1.4-1)}\right\}$$

$$= 0.0701181$$

The entropy $s_1$ upstream of the shock is 7.0032159 kJ kg$^{-1}$ K$^{-1}$ and the specific gas constant for the International Standard Atmosphere $R_{Air}$ is 0.28705287 kJ kg$^{-1}$ K$^{-1}$, so the entropy $s_2$ immediately behind the shock is 7.0233435 kJ kg$^{-1}$ K$^{-1}$.

The air speed $v_2$ immediately behind the shock is

$$v_2 = M_2 a_2 = 0.8656143 \times 338.35934 = 292.88868 \text{ m s}^{-1}$$

The radial velocity component $v_{r,2}$ is

$$v_{r,2} = v_2 \cos(\beta - \alpha) = 292.88868 \cos(68.835 - 14.425583) = 170.45809 \text{ m s}^{-1}$$

and the angular velocity component $v_{\theta,2}$ is

$$v_{\theta,2} = v_2 \sin(\beta - \alpha) = 292.88868 \sin(68.835 - 14.425583) = -238.17603 \text{ m s}^{-1}$$

The reference velocity $v_{max}$ is given by Eq. (1.2),

$$\frac{v_{max}^2}{2} = h_2 + \frac{v_2^2}{2} = 286{,}022.45 + \frac{292.88868^2}{2} = 328{,}914.34 \text{ m}^2 \text{ s}^{-2}$$

so $v_{max}$ is 811.06638 m s$^{-1}$.

The dimensionless velocity components $v'_r$ and $v'_\theta$ are.

$$v'_r = \frac{v_{r,2}}{v_{max}} = \frac{170.45809}{811.06638} = 0.2101654$$

and

$$v'_\theta = \frac{v_{\theta,2}}{v_{max}} = \frac{-238.17603}{811.06638} = -0.2936579$$

## 4.3 Application of the RK4 method

The initial conditions for the RK4 method are $x = 1.2013974$ radians ($\theta = 68.835°$), $y$ $(= v'_r) =$ $0.2101654$ and $z$ $(= v'_\theta) = -0.2936579$. We must integrate Eqs. (4.1) and (4.2) numerically in the negative $x$ direction, and we will choose a step size $h$ of $-0.01$ rad ($-0.5729578°$). When applied to a system of two first-order ordinary differential equations, the RK4 method can be written as

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \quad (4.3a)$$

$$z_{i+1} = z_i + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)h \quad (4.3b)$$

where

$$k_1 = f(x_i, y_i, z_i) \quad (4.3c)$$

$$l_1 = g(x_i, y_i, z_i) \quad (4.3d)$$

$$k_2 = f(x_i + ½h, y_i + ½hk_1, z_i + ½hl_1) \quad (4.3e)$$

$$l_2 = g(x_i + ½h, y_i + ½hk_1, z_i + ½hl_1) \quad (4.3f)$$

$$k_3 = f(x_i + ½h, y_i + ½hk_2, z_i + ½hl_2) \quad (4.3g)$$

$$l_3 = g(x_i + ½h, y_i + ½hk_2, z_i + ½hl_2) \quad (4.3h)$$

$$k_4 = f(x_i + h, y_i + hk_3, z_i + hl_3) \quad (4.3i)$$

$$l_4 = g(x_i + h, y_i + hk_3, z_i + hl_3) \quad (4.3j)$$

and where

$$f(x_i, y_i, z_i) = z_i \quad (4.3k)$$

$$g(x_i, y_i, z_i) = \frac{z_i^2 y_i - 0.2\left[1 - y_i^2 - z_i^2\right]\left[2y_i + z_i \cot x_i\right]}{0.2\left[1 - y_i^2 - z_i^2\right] - z_i^2} \quad (4.3l)$$

Applying the initial conditions, $y_0 = 0.2101654$ and $z_0 = -0.2936579$ at $x_0 = 1.2013974$, to Eqs. (4.3c) to (4.3j) gives

$$k_1 = f(x_0, y_0, z_0) = z_0 = -0.2936579$$

$$l_1 = g(x_0, y_0, z_0) = \frac{z_0^2 y_0 - 0.2[1 - y_0^2 - z_0^2][2y_0 + z_0 \cot x_0]}{0.2[1 - y_0^2 - z_0^2] - z_0^2}$$

$$= \frac{(-0.2936579)^2(0.2101654) - 0.2[1 - 0.2101654^2 - (-0.2936579)^2][2(0.2101654) + (-0.2936579)\cot 1.2013974]}{0.2[1 - 0.2101654^2 - (-0.2936579)^2] - (-0.2936579)^2}$$

$$= \frac{0.0181236 - 0.2 \times 0.8695956 \times 0.3066347}{0.2 \times 0.8695956 - 0.086235} = -0.4015097$$

$$x_0 + ½h = 1.2013974 + ½(-0.01) = 1.1963974$$

$$y_0 + ½hk_1 = 0.2101654 + ½(-0.01)(-0.2936579) = 0.2116337$$

$$z_0 + ½hl_1 = -0.2936579 + ½(-0.01)(-0.4015097) = -0.2916503$$

$$k_2 = f(x_0 + h, y_0 + ½hk_1, z_0 + ½hl_1) = (z_0 + ½hl_1) = -0.2916503$$

$$l_2 = g(x_0 + h, y_0 + ½hk_1, z_0 + ½hl_1)$$

$$= \frac{(z_0 + ½hl_1)^2(y_0 + ½hk_1)}{0.2[1 - (y_0 + ½hk_1)^2 - (z_0 + ½hl_1)^2] - (z_0 + ½hl_1)^2}$$

$$- \frac{0.2[1 - (y_0 + ½hk_1)^2 - (z_0 + ½hl_1)^2][2(y_0 + ½hk_1) + (z_0 + ½hl_1)\cot(x_0 + ½h)]}{0.2[1 - (y_0 + ½hk_1)^2 - (z_0 + ½hl_1)^2] - (z_0 + ½hl_1)^2}$$

$$= \frac{(-0.2916503)^2(0.2116337)}{0.2[1 - (0.2116337)^2 - (-0.2916503)^2] - (-0.2916503)^2}$$

$$- \frac{0.2[1 - (0.2116337)^2 - (-0.2916503)^2][2(0.2116337) + (-0.2916503)\cot(1.1963974)]}{0.2[1 - (0.2116337)^2 - (-0.2916503)^2] - (-0.2916503)^2}$$

$$= \frac{0.0180015 - 0.2 \times 0.8701513 \times 0.3086685}{0.0889704} = -0.4014384$$

$$y_0 + ½hk_2 = 0.2101654 + ½(-0.01)(-0.2916503) = 0.2116237$$

$$z_0 + ½hl_2 = -0.2936579 + ½(-0.01)(-0.4014384) = -0.2916507$$

$$k_3 = f(x_0 + ½h, y_0 + ½hk_2, z_0 + ½hl_2) = (z_0 + ½hl_2) = -0.2916507$$

$$l_3 = g(x_0 + ½h, y_0 + ½hk_2, z_0 + ½hl_2)$$

$$= \frac{(z_0 + ½hl_2)^2(y_0 + ½hk_2)}{0.2[1 - (y_0 + ½hk_2)^2 - (z_0 + ½hl_2)^2] - (z_0 + ½hl_2)^2}$$

$$- \frac{0.2[1 - (y_0 + ½hk_2)^2 - (z_0 + ½hl_2)^2][2(y_0 + ½hk_2) + (z_0 + ½hl_2)\cot(x_0 + ½h)]}{0.2[1 - (y_0 + ½hk_2)^2 - (z_0 + ½hl_2)^2] - (z_0 + ½hl_2)^2}$$

$$= \frac{(-0.2916507)^2(0.2116237)}{0.2[1 - (0.2116237)^2 - (-0.2916507)^2] - (-0.2916507)^2}$$

$$- \frac{0.2[1 - (0.2116237)^2 - (-0.2916507)^2][2(0.2116237) + (-0.2916507)\cot(1.1963974)]}{0.2[1 - (0.2116237)^2 - (-0.2916507)^2] - (-0.2916507)^2}$$

$$= \frac{0.0180007 - 0.2 \times 0.8701553 \times 0.3086483}{0.0889709} = -0.4014081$$

$$x_0 + h = 1.2013974 + (-0.01) = 1.1913974$$

$$y_0 + hk_3 = 0.2101654 + (-0.01)(-0.2916507) = 0.2130819$$

$$z_0 + hl_3 = -0.2936579 + (-0.01)(-0.4014081) = -0.2896438$$

$$k_4 = f(x_0 + h, y_0 + hk_3, z_0 + hl_3) = (z_0 + hl_3) = -0.2896438$$

$$l_4 = g(x_0 + h, y_0 + hk_3, z_0 + hl_3)$$

$$= \frac{(z_0 + hl_3)^2(y_0 + hk_3)}{0.2[1 - (y_0 + hk_3)^2 - (z_0 + hl_3)^2] - (z_0 + hl_3)^2}$$

$$- \frac{0.2[1 - (y_0 + hk_3)^2 - (z_0 + hl_3)^2][2(y_0 + hk_3) + (z_0 + hl_3)\cot(x_0 + h)]}{0.2[1 - (y_0 + hk_3)^2 - (z_0 + hl_3)^2] - (z_0 + hl_3)^2}$$

$$= \frac{(-0.2896438)^2(0.2130819)}{0.2[1 - (0.2130819)^2 - (-0.2896438)^2] - (-0.2896438)^2}$$

$$- \frac{0.2[1 - (0.2130819)^2 - (-0.2896438)^2][2(0.2130819) + (-0.2896438)\cot(1.1913974)]}{0.2[1 - (0.2130819)^2 - (-0.2896438)^2] - (-0.2896438)^2}$$

$$= \frac{0.0178762 - 0.2 \times 0.8707026 \times 0.3106782}{0.2 \times 0.8707026 - 0.0838935} = -0.4014037$$

Substituting $k_1$, $l_1$, $k_2$, $l_2$, $k_3$, $l_3$, $k_4$ and $l_4$ into (4.3a) and (4.3b) gives

$$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]h$$

$$= 0.2101654 + \frac{1}{6}[-0.2936579 + 2(-0.2916503) + 2(-0.2916507) + (-0.2896438)](-0.01)$$

$$= 0.2130819$$

$$z_1 = z_0 + \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]h$$

$$= -0.2936579 + \frac{1}{6}[-0.4015097 + 2(-0.4014384) + 2(-0.4014081) + (-0.4014037)](-0.01)$$

$$= -0.2896435$$

We now have a new set of conditions, $y_1 = 0.2130819$ and $z_1 = -0.2896435$ at $x_1 = 1.1913974$ rad. We simply repeat the preceding steps as many times as necessary until $z$ reaches zero or becomes positive. Table 3 shows that $z$ becomes positive when $x = 0.5713974$ rad ($\theta = 32.7386°$), which is greater than the semi-vertex angle $\lambda$. Consequently, we must reduce the value of $x_0$ (the wave angle, $\beta$), recalculate the shock properties from the equations for an oblique shock, and repeat the RK4 calculation.

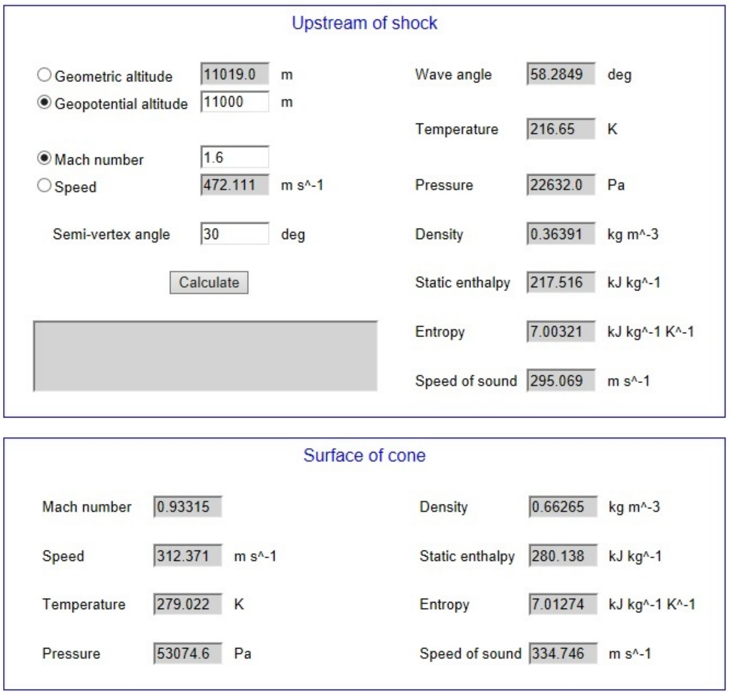
To illustrate the application of the RK4 method, we chose a step size $h$ of $-0.01$ rad ($-0.5729578°$). In the web application the step size $h$ is $-0.000087266$ rad ($-0.005°$) and the increment in the wave angle $\beta$ is $-0.00087266$ rad ($-0.05°$).

**Table 4  RK4 solution**

| $i$ | $x_i$ | $y_i$ | $z_i$ | $i$ | $x_i$ | $y_i$ | $z_i$ |
|---|---|---|---|---|---|---|---|
| 0 | 1.2013974 | 0.2101654 | −0.2936579 | 40 | 0.8013974 | 0.2941445 | −0.1203417 |
| 1 | 1.1913974 | 0.2130819 | −0.2896435 | 41 | 0.7913974 | 0.2953234 | −0.1154499 |
| 2 | 1.1813974 | 0.2159827 | −0.2856290 | 42 | 0.7813974 | 0.2964533 | −0.1105198 |
| 3 | 1.1713974 | 0.2187945 | −0.2816118 | 43 | 0.7713974 | 0.2975337 | −0.1055501 |
| 4 | 1.1613974 | 0.2215905 | −0.2775899 | 44 | 0.7613974 | 0.2985642 | −0.1005397 |
| 5 | 1.1513974 | 0.2243463 | −0.2735614 | 45 | 0.7513974 | 0.2995444 | −0.0954873 |
| 6 | 1.1413974 | 0.2270617 | −0.2695246 | 46 | 0.7413974 | 0.3004738 | −0.0903914 |
| 7 | 1.1313974 | 0.2297367 | −0.2654778 | 47 | 0.7313974 | 0.3013520 | −0.0852506 |
| 8 | 1.1213974 | 0.2323712 | −0.2614198 | 48 | 0.7213974 | 0.3021786 | −0.0800634 |
| 9 | 1.1113974 | 0.2349651 | −0.2573491 | 49 | 0.7113974 | 0.3029531 | −0.0748279 |
| 10 | 1.1013974 | 0.2375181 | −0.2532646 | 50 | 0.7013974 | 0.3036750 | −0.0695425 |
| 11 | 1.0913974 | 0.2400303 | −0.2491651 | 51 | 0.6913974 | 0.3043438 | −0.0642050 |
| 12 | 1.0813974 | 0.2425014 | −0.2450498 | 52 | 0.6813974 | 0.3049590 | −0.0588135 |
| 13 | 1.0713974 | 0.2449312 | −0.2409175 | 53 | 0.6713974 | 0.3055199 | −0.0533656 |
| 14 | 1.0613974 | 0.2473197 | −0.2367675 | 54 | 0.6613974 | 0.3060261 | −0.0478589 |
| 15 | 1.0513974 | 0.2496665 | −0.2325988 | 55 | 0.6513974 | 0.3064769 | −0.0422906 |
| 16 | 1.0413974 | 0.2519716 | −0.2284107 | 56 | 0.6413974 | 0.3068717 | −0.0366580 |
| 17 | 1.0313974 | 0.2542347 | −0.2242025 | 57 | 0.6313974 | 0.3072098 | −0.0309580 |
| 18 | 1.0213974 | 0.2564556 | −0.2199733 | 58 | 0.6213974 | 0.3074906 | −0.0251870 |
| 19 | 1.0113974 | 0.2586341 | −0.2157226 | 59 | 0.6113974 | 0.3077133 | −0.0193414 |
| 20 | 1.0013974 | 0.2607700 | −0.2114497 | 60 | 0.6013974 | 0.3078772 | −0.0134173 |
| 21 | 0.9913974 | 0.2628630 | −0.2071538 | 61 | 0.5913974 | 0.3079814 | −0.0074101 |
| 22 | 0.9813974 | 0.2649130 | −0.2028345 | 62 | 0.5813974 | 0.3080251 | −0.0013152 |
| 23 | 0.9713974 | 0.2669196 | −0.1984910 | 63 | 0.5713974 | 0.3080074 | 0.0048727 |
| 24 | 0.9613974 | 0.2688827 | −0.1941227 | | | | |
| 25 | 0.9513974 | 0.2708020 | −0.1897291 | | | | |
| 26 | 0.9413974 | 0.2726772 | −0.1853095 | | | | |
| 27 | 0.9313974 | 0.2745081 | −0.1808634 | | | | |
| 28 | 0.9213974 | 0.2762944 | −0.1763900 | | | | |
| 29 | 0.9113974 | 0.2780358 | −0.1718889 | | | | |
| 30 | 0.9013974 | 0.2797320 | −0.1673592 | | | | |
| 31 | 0.8913974 | 0.2813829 | −0.1628005 | | | | |
| 32 | 0.8813974 | 0.2829880 | −0.1582120 | | | | |
| 33 | 0.8713974 | 0.2845470 | −0.1535929 | | | | |
| 34 | 0.8613974 | 0.2860597 | −0.1489427 | | | | |
| 35 | 0.8513974 | 0.2875260 | −0.1442606 | | | | |
| 36 | 0.8413974 | 0.2889448 | −0.1395457 | | | | |
| 37 | 0.8313974 | 0.2903166 | −0.1347973 | | | | |
| 38 | 0.8213974 | 0.2916406 | −0.1300144 | | | | |
| 39 | 0.8113974 | 0.2929167 | −0.1251962 | | | | |

Figure 11 shows the output from the web application for the upstream conditions used to illustrate the RK4 method (geopotential altitude of 11,000 m, upstream Mach number $M_1$ of 1.6, and cone semi-vertex angle $\lambda$ of 30°). The calculated wave angle $\beta$ is 58.2849°, and this value agrees with the plot of wave angle $\beta$ against semi-vertex angle $\lambda$ for different upstream Mach numbers $M_1$ in Figure 2. The shock wave gives rise to an increase in temperature, pressure and density, and the flow at the surface of the cone is subsonic.

**Figure 11  Output from the web application**



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